Symplectic Particle-In-Cell Method for 1-Dimensional Vlasov-Possion Plasma System

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Particle-In-Cell (PIC) Method - Scheme

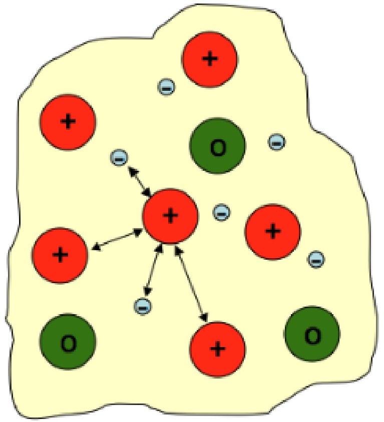
- Particle – Mesh coupling method for solving the equations of motion in plasma
- Discretization of the particle distribution (Klimotovich approach)

$$f_s \approx f_{s,h}(x, v, t) = \sum w_i \delta(x - X_{s,i}(t)\delta(v - V_{s,i}(t))$$

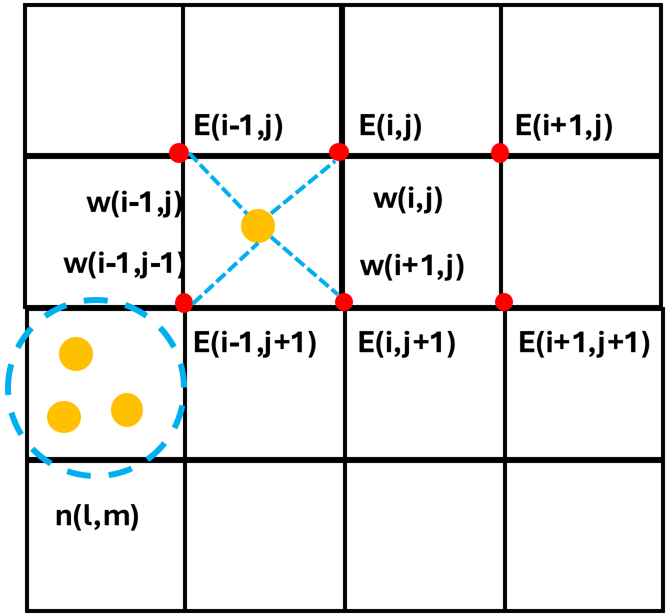
- Separation of the particle coordinate and mesh grid for EM fields
  - E and B fields: computed on the coarse grid
  - X and V: computed on the particle coordinate

$$\boldsymbol{F(x_k) = \sum k \frac{q_i q_j}{\eta_{ij}^2} \widehat{\eta_{ij}}}$$

High computational cost!



Mesh grid



Count particle number for each grid

Field solver (Mesh grid)

$$\boldsymbol{\nabla^2 \phi = qn} \qquad \boldsymbol{E = -\nabla \phi}$$

Interpolation (Mesh → particle)

$$\boldsymbol{E(x_k) = \sum w_{i,j} E_{ij}}$$

$$\boldsymbol{m \frac{dv_k}{dt} = qE(x_k, t)}$$

Particle mover

Particle-In-Cell (PIC) Method - Implementation

- **Field solver**: Solve Poisson equation on the mesh coordinate (coarse grid)
  - Discretization of the Laplacian: Finite Difference Method (FDM) with 2nd order
  - Linear equation solver: Thomas algorithm (Gaussian elimination) with Sherman-Morrison formula for periodic system

$$\nabla^2 \boldsymbol{\phi} = \boldsymbol{qn} \qquad \boldsymbol{E} = -\nabla \boldsymbol{\phi}$$

$$\Rightarrow \quad L = \begin{bmatrix} -2\;1 & \cdots & 0 & 1 \\ \vdots & \ddots & & \vdots \\ 1\;0 & \cdots & 1 & -2 \end{bmatrix} \qquad G = \begin{bmatrix} 0\;1 & \cdots & -1 \\ \vdots & \ddots & \vdots \\ -1\;0 & \cdots & 1\;0 \end{bmatrix}$$

Periodic Tridiagonal matrix

$$\Rightarrow \quad A = \begin{bmatrix} b_1 & \cdots & a_1 \\ \vdots & \ddots & \vdots \\ c_n & \cdots & b_n \end{bmatrix} \qquad u = [\gamma, 0, \dots, 0, c_n]^T \qquad \mathrm{v} = [1, 0, \dots, 0, a_1/\gamma]^T$$

$$B = \begin{bmatrix} b_1 - \gamma & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & b_n - \dfrac{c_n a_1}{\gamma} \end{bmatrix}$$

Modified Tridiagonal matrix

Solve it with Gauss Elimination

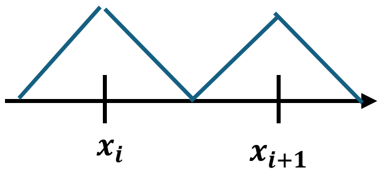
$$By = b \qquad Bq = u$$

$$x = A^{-1}b = (B + uv^T)^{-1}b = y - \frac{qv^Ty}{1 + v^Tq}$$

Then, compute x for Ax = b

- **Interpolation**: Convert E-field of the mesh grid to particle's coordinate
  - Triangular Shaped Clould (TSC) scheme: 2nd order weighting scheme

$$\boldsymbol{E}(\boldsymbol{x}) = \sum \boldsymbol{E}_i^{mesh}(\boldsymbol{x}_i)\boldsymbol{S}(\boldsymbol{x} - \boldsymbol{x}_i) \qquad \boldsymbol{S}(\boldsymbol{x} - \boldsymbol{x}_i) \coloneqq \begin{cases} \boldsymbol{1} - \dfrac{|\boldsymbol{x} - \boldsymbol{x}_i|}{\boldsymbol{\Delta x}} \\ \boldsymbol{0} \end{cases}$$



- **Particle mover**: Time integration of the particle's motion from Hamilton's equation $\dot{z} = J\nabla_z H(z)$
  - Symplectic time integrator: For canonical coordinates (q,p), the time evolution of Hailton's equation conserving symplectic 2-form

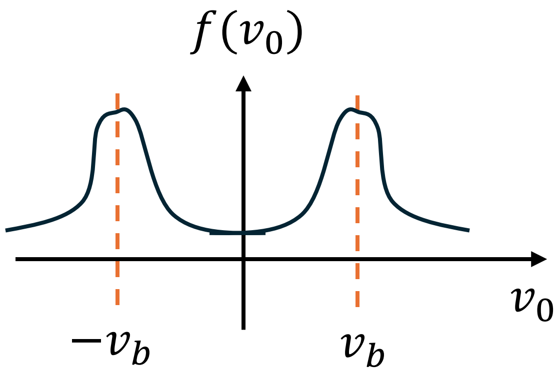
$$H(q,p) = T(p) + V(q)$$

$$\begin{pmatrix} \dot{q} \\ \dot{p} \end{pmatrix} = \begin{pmatrix} \partial T(p)/\partial p \\ -\partial V(q)/\partial q \end{pmatrix} \longrightarrow \begin{pmatrix} q_{i+1} \\ p_{i+1} \end{pmatrix} = \begin{pmatrix} q_i + c_i p_i \Delta t \\ p_i + d_i F(q_i) \Delta \mathrm{t} \end{pmatrix}$$

Symplectic Euler method	$c_1 = 0\; c_2 = 1 \quad d_1 = 1\; d_2 = 0$	Conserve H with 1st order
Leapfrog method	$c_1 = 1\; c_2 = 0 \quad d_1 = d_2 = \frac{1}{2}$	Conserve H with 2nd order

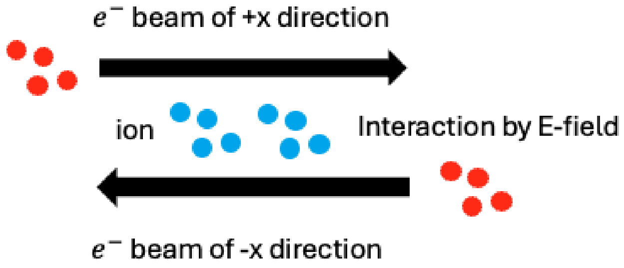
Application to PIC simulation

- Case 1. Two-stream instability

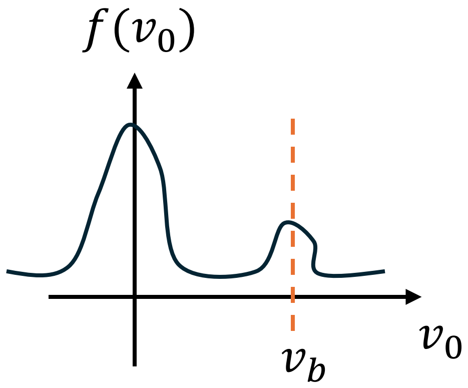


$$f(v_0) = \frac{1}{2\sqrt{2\pi}} e^{-\frac{1}{2}(v_0 - v_b)^2} + \frac{1}{2\sqrt{2\pi}} e^{-\frac{1}{2}(v_0 + v_b)^2}$$

- Two electron beams with opposite direction in a fixed ion background
- Each beam follows Gaussian distribution with $\bar{v} = \pm v_b$



- Case 2. Bump-on-tail instability



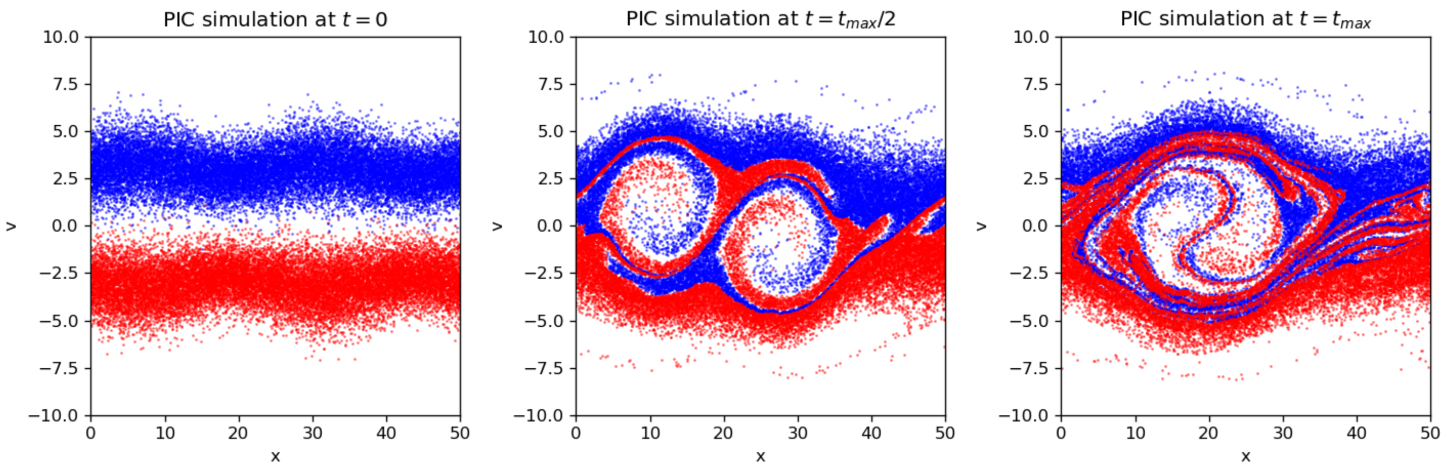
$$f(v_0) = \frac{1}{(1+a)\sqrt{2\pi}} e^{-\frac{1}{2}v_0^2} + \frac{a}{(1+a)\sqrt{2\pi}\sigma} e^{-\frac{1}{2\sigma^2}(v_0 - v_b)^2}$$

- Thermalized electron distribution with injected high energy electron
- The high energy e^- beam with v_0 makes bump on its tail of the distribution

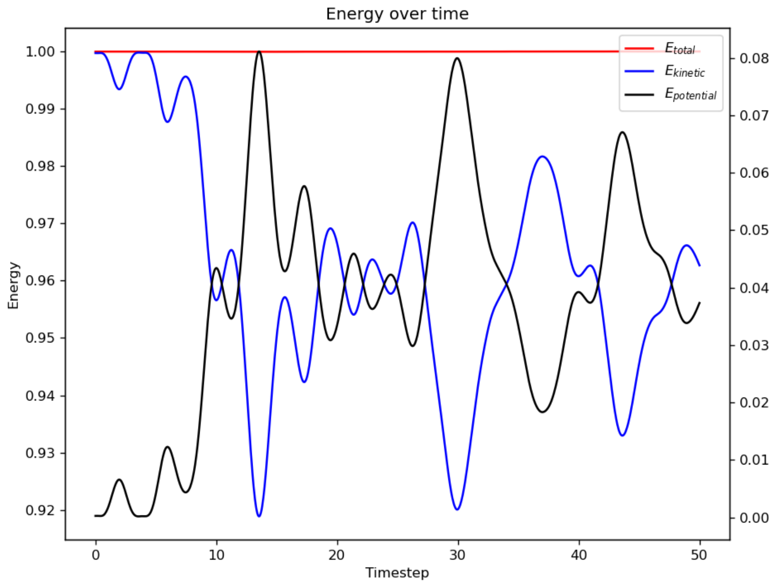
- Setting for numerical simulation (common)
  - Sine wave perturbation : $v(x, t = 0) = v_0(1 + A \sin 2\pi n_{mode}\, x/L)$
  - $v_b = 3.0,\ a = 0.2,\ \sigma = 0.5, A = 0.02,\ n_{mode} = 5,\ N = 40000, N_{mesh} = 1000, L = 50, dt = 0.05, t_{max} = 50.0$
  - Spatial resolution condition: $\Delta x < 3.4\, \lambda_{Debye}$
  - Temporal resolution condition: $\Delta t < 2\omega_{pe}^{-1}$

Numerical Results: Two-Stream Instability

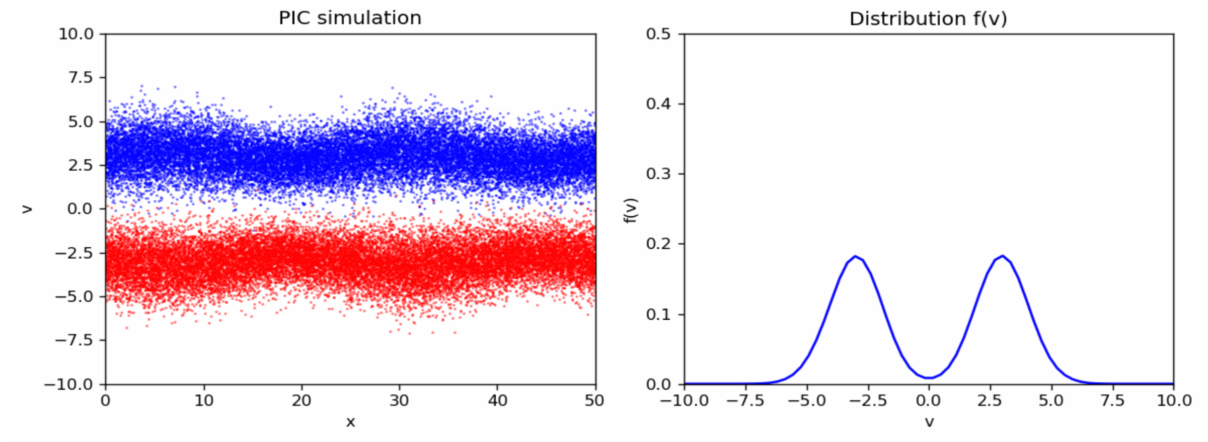
- Simulation result



- System Hamiltonian over time



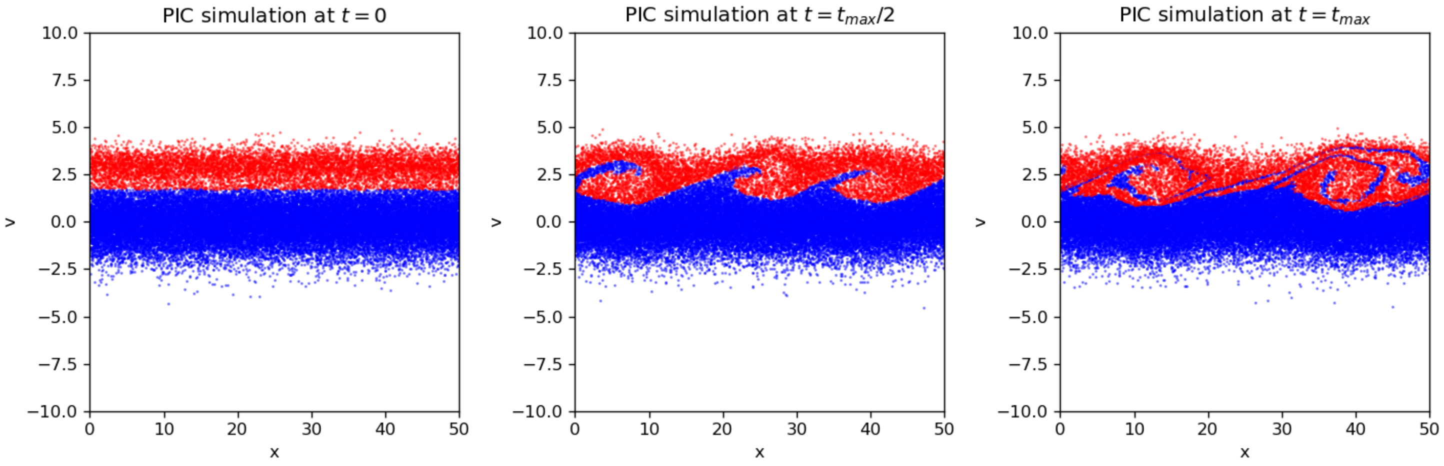
- Time evolution of the PIC simulation and velocity distribution



- Energy conserved during the simulation by symplectic integrator (Leapfrog)
- Separatrix induced by sinusoidal perturbation observed
- Velocity distribution reaches to new equilibrium

Numerical Results: Bump-On-Tail Instability

- Simulation result



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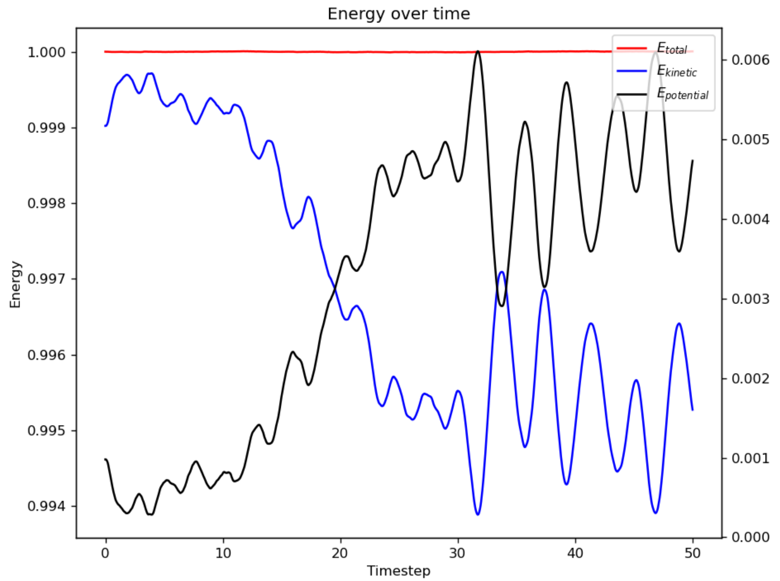
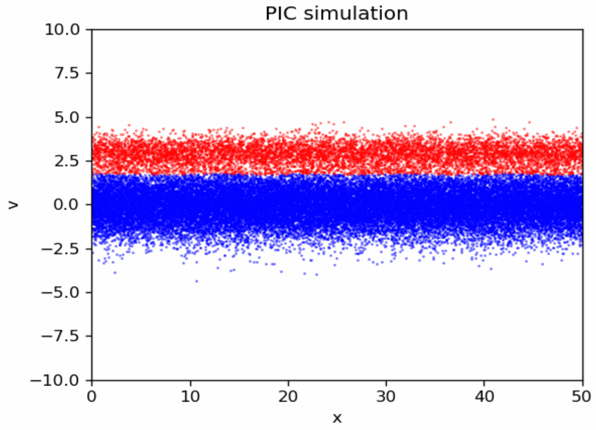
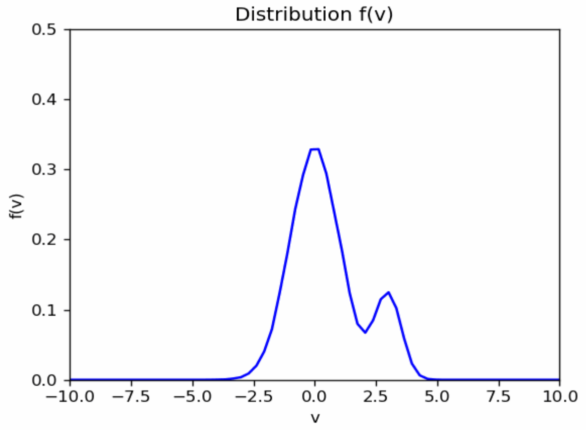


- Time evolution of the PIC simulation and velocity distribution





- Energy also conserved during the simulation by symplectic integrator (Leapfrog)
- High energy electron with sinusoidal perturbation induces the instability
- Quasi-linear diffusion induced by the velocity space observed and results in the plateau in the tails of the distribution